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Discrete Convexes

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ABSTRACT

In this report we will introduce an algorithm that will be an alternative for algorithms such as *flood-fill*, *boundary-fill* and *tint-fill* algorithms. This is needed since these algorithms are recursive ones. This means that these algorithms may cause stack overflow when memory space is limited.

Introduction

In this report we will study 2D and 3D graphics in the discrete plane with some algorithms traversing a convex region such as *flood-fill*, *boundary-fill* and *tint-fill* algorithms[16][17] or even using Supercovers[4][12].

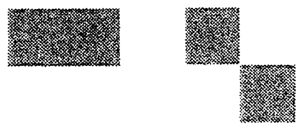
1 Geometrical Primitives

In this subsection some definitions and facts from discrete geometry will be given together with some examples. The following denotations will be used throughout this report. $\mathbb{R}$ will denote the set of real numbers, $\mathbb{N}$ the set of natural numbers, $\mathbb{Z}$ the set of integers. $\mathbb{Z}^2$ will denote the discrete cartesian plane, all ordered pairs in the form $\{(x, y) \,|x, y \in \mathbb{Z}\}$. $[a]$ denotes the greatest integer less than or equal to a.

1.1 Pixels

Definition 1 *The discrete 2D cartesian plane consists of unit squares, called pixels which are centered on the integer points of the 2D cartesian coordinate system.*

Definition 2 *2D adjacency: Two pixels are 4-adjacent if they have a common edge and 8-adjacent if they have a common edge or vertex. Pixels adjacent to a pixel p are said to be in the neighbourhood of p.*



4-adjacent pixels 8-adjacent pixels

Definition 3 *A sequence of pixels is a k-path (k=4,8) if every two consecutive pixels along the sequence are k-adjacent.*

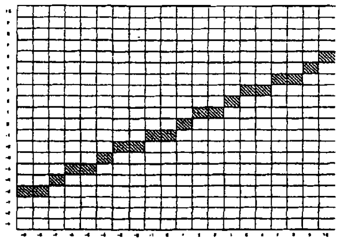
Definition 4 *2D connectivity: Two pixels are k-connected (k=4,8) if there exists a k-path between them. A set of pixels are connected if there exists at least an 8-path between every two pixels. Otherwise it is disconnected.*

1.2 2D Lines

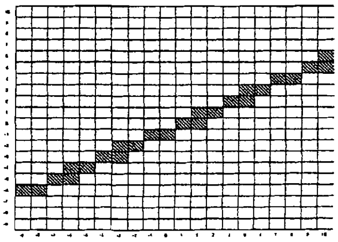
Definition 5 *A 2D discrete line is a set of pixels* $L(a, b, \mu, \omega) = \{(x, y) \in \mathbb{Z}^2 \,|0 \leq ax + by + \mu < \omega\}$

where $a, b, \mu \in \mathbb{Z}$ and $\omega \in \mathbb{N}$. ω is called the arithmetic thickness of the discrete line and μ is called the internal translation constant.[5][15]

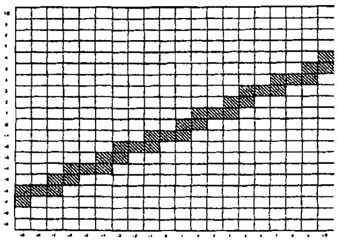
Definition 6 *A discrete line $L(a, b, \mu, \omega)$ is 4-connected (standard) if $\omega = |a| + |b|$, 8-connected (naive if $\omega = \max\{|a|, |b|\}$, *-connected if $\max\{|a|, |b|\} < \omega < |a| + |b|$ and thick if $\omega \geq |a| + |b|$*



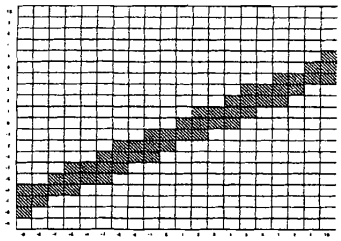
$0 \leq 5x - 8y - 1 < 8$ (naive)



$0 \leq 5x - 8y - 1 < 10$ (*-connected)

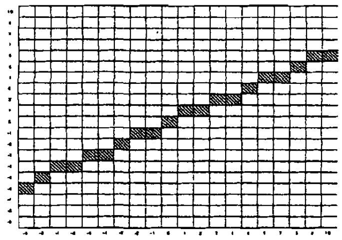


$0 \leq 5x - 8y - 1 < 13$ (standard)



$0 \leq 5x - 8y - 1 < 20$ (thick)

Definition 7 *Let $L(a, b, \mu, \omega)$ be a naive line passing through $O(0,0)$ such that $\mu = \left[\frac{\omega}{2}\right]$ then this line is a BRESENHAM line. A Bresenham line has the following property: Let $\mathcal{L}$ be a straight line in the plane and L_B be the related Bresenham line. Suppose L_B is functional on the x-axis and let for some $x \in \mathbb{Z}$, $(x, y) \in L_B$, $(x, \bar{y}) \in \mathcal{L}$ then $|y - \bar{y}| \leq 1/2$.*



$0 \leq 5x - 8y + 3 < 8$ (Bresenham)

1.3 Circles

A circle passing through the origin $O(0,0)$ can be expressed in the continous plane as follows in the continous plane

$$x^2 + y^2 = r^2$$

or

$$x = r\cos\theta, y = r\sin\theta$$

where r is the radius of the circle and θ is the angle between the x-axis and the line segment from the origin to (x, y).

So for every x we can calculate y as $y = \pm\sqrt{r^2 - x^2}$ which can be done by evaluating the square root function which also leads to inefficiency in the algorithm or in using polar coordinates this leads to evaluating cos or sin functions.

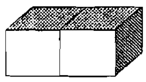
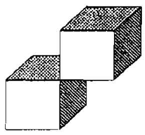
In evaluating the points of a circle the symmetry of a circle is used.[1]

1.4 3D Lines

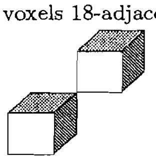
Let Oxy,Oyz and Oxz denote the coordinate planes in the three dimensional Cartesian coordinate system and $\mathbb{Z}^3$ denote the set of ordered triples $\{(x, y, z) \,|x, y, z \in \mathbb{Z}\}$.

Definition 8 *The discrete coordinate space consists of unit cubes called voxels which are centered on the integer points of the three dimensional Cartesian coordinate system.*

Definition 9 *3D Adjacency: Two voxels are 6-adjacent if they share a common face. Voxels are 18-adjacent if they share a common face or edge. They are 26-adjacent if they share a common face, edge or vertex.*

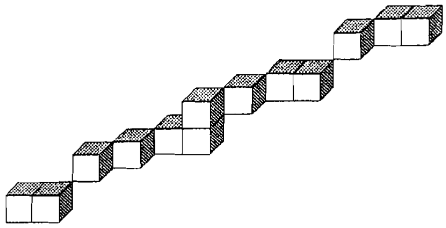



6-adjacent voxels 18-adjacent voxels



26-adjacent voxels

Definition 10 *A 3D discrete line is a set of voxels* $L(a, b, c, \mu, \mu', e, e') = \{(x, y, z) \in \mathbb{Z}^3 | \mu \leq cx - az < \mu + e$ *and* $\mu' \leq bx - ay < \mu' + e'\}$ *where the direction vector is* (a, b, c) *and* $a \geq b \geq c$*.if* $e = e' = a$ *then this line is a naive line.*



A 3D line

Definition 11 *Given a line* $L(a, b, c, \mu, \mu', e, e')$ *if* $e \geq a + c$ *and* $e' \geq a + b$ *then* L *is 6-connected. If* $e \geq a + c$ *and* $a \leq e' < a + b$ *or* $e' \geq a + b$ *and* $a \leq e < a + c$ *is 18-connected. If* $a \leq e < a + c$ *and* $a \leq e' < a + b$ *then* L *is 26-connected. If* $e < a$ *or* $e' < a$ *it is disconnected.*

1.5 Planes

Definition 12 *A discrete plane is a set of voxels* $P(a, b, c, \mu, \omega) = \{(x, y, z \in \mathbb{Z}^3 | 0 \leq ax + by + cz + \mu \leq \omega\}$ *where* $a, b, c, \mu \in \mathbb{Z}$ *and* $\omega \in \mathbb{N}$. ω *is the arithmetical thickness of the plane and* μ *the internal translation constant. The vector* (a, b, c) *is the normal vector of the plane.[13][8][10]*

Definition 13 *3D Adjacency: Two voxels are 6-adjacent if they share a common face, 18-adjacent if they share a common face or edge and 26-adjacent if they share a common face, edge or vertex*

Definition 14 *The plane* $P(a, b, c, \mu, \omega)$ *has k-tunnel* ($k = 6, 18, 26$) *if there exists two k-adajcent voxels* $A(x_A, y_A, z_A)$ *and* $B(x_B, y_B, z_B)$ *such that* $ax_A + by_A + cz_A + \mu < 0$ *and* $ax_B + by_B + cz_B + \mu \geq \omega$

1.6 Triangles and Meshes

Triangles are used because of the following properties[6][7][8][9]

1. **Geometric properties analogous to ones of a continous triangle:**

i) A triangle is a part of a discrete plane which is unambiguously defined.

ii) A triangle is fully defined by its three vertices.

2. **Geometrical properties useful for obtaining a good approximation to the continous plane:**

   i) A single triangle:

   a) The triangle is as thin as possible.

   b) The triangle is an at least 26-connected set.

   ii) A mesh of triangles

   a) The union of two triangles with a common edge is at least 6-tunnel free.

3. **Algorithmical and analytical properties**

   i) The triangle can be described analytically by formulae.

   ii) The finite set of points of a triangle can be generated by efficient algorithms.

1.7 Supercover Approach

A supercover of a continous object is a set of pixels or voxels which are intersected by the object. The supercover approach has many advantages.[4][12]

- The polyhedra obtained through this method are tunnel-free, which is important for the ray-tracing applied in a later stage of the visualization pipeline.
- They have good properties under set operations. For example the supercover of the union of two objects is the union of the seperate supercovers of these objects which is interesting for discrete modeling.
- The supercover of an object is not dependent of the order in which the parts of the object are discretized. The supercover of the line from point A to point B is identical to the supercover of the line from point B to point A.
- The supercovers of rational points, lines, line segment, triangles, triangles sharing a common edge, both in 2D and 3D, are characterized by discrete analytical descriptions, leading to fast integer-based generation and localization algorithms.

However sometimes a supercovers are not the thinnest discretization.

2 Half Planes

Definition 15 *A half plane is an unbounded region which is above or below a line. A line can be used to define two half planes.*

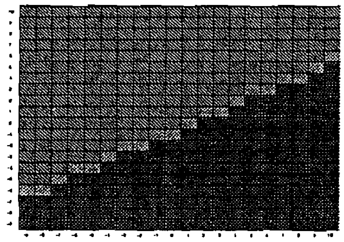
Definition 16 *Given a line* $L(a, b, \mu, \omega) = \{(x, y) \in \mathbb{Z}^2 \,|\, 0 \leq ax + by + \mu < \omega\}$ *the two half planes can be defined as follows:*

$$\begin{aligned} ax + by + \mu &\geq \omega \\ ax + by + \mu &< 0 \end{aligned}$$

without the line or by

$$\begin{aligned} ax + by + \mu &\geq 0 \\ ax + by + \mu &< \omega \end{aligned}$$

with the line.



Half-plane (not including the line)

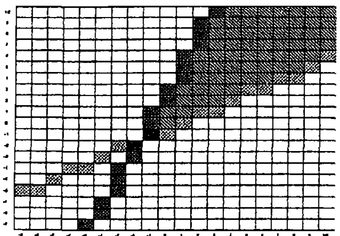
If L is naive or *-connnected the two planes will be 8-connected that is there exists an 8-path between the two half planes else the two half planes will be disconnected.

3 Convexes

A convex is a region in which every pair of points in that region can be joined by a straight line in such a way that the line never enters the exterior of that region.[16][17]. Convexes are used in algorithms as a clipping region

4 Angular Sections

Definition 17 *An angular section is defined by the intersection of two half planes.*



Angular section of two lines

4.1 Analysis of Angular Sections

Two lines define four angular sections.

$$a_1x + b_1y + \mu_1 \geq \omega_1 \text{ and } a_2x + b_2y + \mu_2 \geq \omega_2 \quad \text{(I)}$$

$$a_1x + b_1y + \mu_1 < 0 \text{ and } a_2x + b_2y + \mu_2 < 0 \quad \text{(II)}$$

$$a_1x + b_1y + \mu_1 \geq \omega_1 \text{ and } a_2x + b_2y + \mu_2 < 0 \quad \text{(III)}$$

$$a_1x + b_1y + \mu_1 < 0 \text{ and } a_2x + b_2y + \mu_2 \geq \omega_2 \quad \text{(IV)}$$

Consider region I. This implies

$$\begin{aligned} a_1x + b_1y &\geq \omega_1 - \mu_1 \Rightarrow a_1x \geq \omega_1 - \mu_1 - b_1y \\ a_2x + b_2y &\geq \omega_2 - \mu_2 \Rightarrow a_2x \geq \omega_2 - \mu_2 - b_2y \end{aligned}$$

i) $a_1 > 0$ and $a_2 > 0$ implies

$$\begin{aligned} x &\geq \frac{\omega_1 - \mu_1 - b_1y}{|a_1|} \text{ and } x \geq \frac{\omega_2 - \mu_2 - b_2y}{|a_2|} \\ &\Rightarrow x \geq \max_i \{\frac{\omega_i - \mu_i - b_iy}{|a_i|}\} \end{aligned}$$

ii) $a_1 > 0$ and $a_2 < 0$ implies

$$x \geq \frac{\omega_1 - \mu_1 - b_1y}{|a_1|} \text{ and } x \leq -\frac{\omega_2 - \mu_2 - b_2y}{|a_2|}$$

iii) $a_1 < 0$ and $a_2 > 0$ implies

$$x \leq -\frac{\omega_1 - \mu_1 - b_1x}{|a_1|} \text{ and } x \geq \frac{\omega_2 - \mu_2 - b_2y}{|a_2|}$$

iv) $a_1 < 0$ and $a_2 < 0$ implies

$$x \leq -\frac{\omega_1 - \mu_1 - b_1y}{|a_1|} \text{ and } x \leq -\frac{\omega_2 - \mu_2 - b_2y}{|a_2|}$$

$$\Rightarrow x \leq \min_i \{-\frac{\omega_i - \mu_i - b_iy}{|a_i|}\}$$

Consider region II. This implies

$$\begin{aligned} a_1x + b_1y &< -\mu_1 \Rightarrow a_1x < -\mu_1 - b_1y \\ a_2x + b_2y &< -\mu_2 \Rightarrow a_2x < -\mu_2 - b_2y \end{aligned}$$

i) $a_1 > 0$ and $a_2 > 0$ implies

$$x < -\frac{\mu_1 + b_1y}{|a_1|} \text{ and } x < -\frac{\mu_2 + b_2y}{|a_2|}$$

$$\Rightarrow x < \min_i \{-\frac{\mu_i + b_iy}{|a_i|}\}$$

ii) $a_1 > 0$ and $a_2 < 0$ implies

$$x < -\frac{\mu_1 + b_1y}{|a_1|} \text{ and } x > \frac{\mu_2 + b_2y}{|a_2|}$$

iii) $a_1 < 0$ and $a_2 > 0$ implies

$$x > \frac{\mu_1 + b_1 x}{|a_1|} \text{ and } x < -\frac{\mu_2 + b_2 y}{|a_2|}$$

iv) $a_1 < 0$ and $a_2 < 0$ implies

$$x > \frac{\mu_1 + b_1 y}{|a_1|} \text{ and } x > \frac{\mu_2 + b_2 y}{|a_2|}$$

$$\Rightarrow x > \max_i \{\frac{\mu_i + b_i y}{|a_i|}\}$$

Consider region III. This implies

$$\begin{aligned} a_1 x + b_1 y &\geq \omega_1 - \mu_1 \Rightarrow a_1 x \geq \omega_1 - \mu_1 - b_1 y \\ a_2 x + b_2 y &< -\mu_2 \Rightarrow a_2 x < -\mu_2 - b_2 y \end{aligned}$$

i) $a_1 > 0$ and $a_2 > 0$ implies

$$x \geq \frac{\omega_1 - \mu_1 - b_1 y}{|a_1|} \text{ and } x < -\frac{\mu_2 + b_2 y}{|a_2|}$$

ii) $a_1 > 0$ and $a_2 < 0$ implies

$$x \geq \frac{\omega_1 - \mu_1 - b_1 y}{|a_1|} \text{ and } x > \frac{\mu_2 + b_2 y}{|a_2|}$$

iii) $a_1 < 0$ and $a_2 > 0$ implies

$$x \leq -\frac{\omega_1 - \mu_1 - b_1 y}{|a_1|} \text{ and } x < -\frac{\mu_2 + b_2 y}{|a_2|}$$

iv) $a_1 < 0$ and $a_2 < 0$ implies

$$x \leq -\frac{\omega_1 - \mu_1 - b_1 y}{|a_1|} \text{ and } x > \frac{\mu_2 + b_2 y}{|a_2|}$$

Consider region IV. This implies

$$\begin{aligned} a_1 x + b_1 y &< -\mu_1 \Rightarrow a_1 x < -\mu_1 - b_1 y \\ a_2 x + b_2 y &\geq \omega_2 - \mu_2 \Rightarrow a_2 x \geq \omega_2 - \mu_2 - b_2 y \end{aligned}$$

i) $a_1 > 0$ and $a_2 > 0$ implies

$$x < -\frac{\mu_1 + b_1 y}{|a_1|} \text{ and } x \geq \frac{\omega_2 - \mu_2 - b_2 y}{|a_2|}$$

ii) $a_1 > 0$ and $a_2 < 0$ implies

$$x < -\frac{\mu_1 + b_1 y}{|a_1|} \text{ and } x \leq -\frac{\omega_2 - \mu_2 - b_2 y}{|a_2|}$$

iii) $a_1 < 0$ and $a_2 > 0$ implies

$$x > \frac{\mu_1 + b_1 y}{|a_1|} \text{ and } x \geq \frac{\omega_2 - \mu_2 - b_2 y}{|a_2|}$$

iv) $a_1 < 0$ and $a_2 < 0$ implies

$$x > \frac{\mu_1 + b_1 y}{|a_1|} \text{ and } x \leq -\frac{\omega_2 - \mu_2 - b_2 y}{|a_2|}$$

4.2 Algorithm for Tracing an Angular Section

This algorithm works for region I and can be easily modified to trace the other regions. Assume $a_1, a_2 > 0$ and that in region I the second line is above the first.

Step 1. Input the coefficients of the lines, a_1,b_1,μ_1,ω_1,a_2,b_2,μ_2,ω_2 .

Step 2. i) Let $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$

ii) Initialize x to the minimum value of x such that there exists a pixel in the region.

iii) Let $y_1 = \frac{(w_1 - \mu_1)}{b_1} + m_1 x$ and $y_2 = \frac{(w_2 - \mu_2)}{b_2} + m_2 x$

iv) While $x \leq 10$
  - a) While $y_1 \leq y_2$
    - (i) Display the pixel (x, y)
    - (ii) Increment y by 1
  - b) Increment x by 1
  - c) Increment y_1 by m_1
  - d) Increment y_2 by m_2

5 Conclusion

In this report we have studied the basic concepts in 2D and 3D graphics together with a review of well known algorithms such as *flood-fill* algorithm. Then we have analized angular sections defined by two 2D discrete lines in the discrete planes and have constructed an algorithm to traverse (colour) an angular section.

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